20071102 Instanton counting

○ Definition of instanton

X : oriented 4-manifold

g : Riemannian metric

$* : \Lambda^2 \circlearrowleft$ Hodge star operator

$$*^2 = \mathrm{id}$$

$$\Lambda^2 \cong \Lambda^+ \oplus \Lambda^- \qquad (\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3))$$

$P \longrightarrow X$ G-principal bundle

(G : compact Lie group)

A connection A on P is anti-self-dual (instanton)

$$\overset{\text{def.}}{\Longleftrightarrow} \quad F_A^+ = 0$$

Suppose X : compact

$$\int_X \mathrm{tr}(F_A \wedge F_A) = \int_X \mathrm{tr}(F_A^+ \wedge F_A^+) + \mathrm{tr}(F_A^- \wedge F_A^-)$$

$$= \|F_A^+\|_{L^2}^2 - \|F_A^-\|_{L^2}^2$$ is a topological invariant of P, independent of A

$\therefore$ an anti-self-dual connection is an absolute minimum of the Yang-Mills functional

$$A \mapsto \|F_A\|_{L^2}^2$$

(BPS condition in Sergei's lecture)

$\mathcal{M}(P)$: moduli space of ASD connections on P
= gauge equiv. classes of ASD connections on P

○ $N=2$ SUSY Yang-Mills theory v.s. Donaldson invariants
(Witten, Atiyah-Jeffrey)

$\mathcal{A}$: the space of all connections on P
$\mathcal{G}$: the group of bundle automorphisms of P
$\mathcal{A} \curvearrowleft \mathcal{G}$
$\mathcal{B} = \mathcal{A}/\mathcal{G}$

$\Omega^+(\mathrm{ad}P)$: the space of $\mathrm{ad}P$-valued self-dual 2-forms

We consider it as a vector bundle over $\mathcal{B}$ as $(\Omega^+(\mathrm{ad}P) \times \mathcal{A})/\mathcal{G} \to \mathcal{A}/\mathcal{G}$

Then $A \mapsto F_A^+$ is its section.
zero set = moduli space of ASD connections

[zero set] = [moduli of ASD connections]
$\cap$ is considered as the Poincaré dual of the
$H_*(\mathcal{B})$ the Euler class of the bundle $\Omega^+(\mathrm{ad}P)$.

$$\text{degree} = \text{"dim"}\,\mathcal{B} - \text{"rank"}\,\Omega^+(\mathrm{ad}\,P)$$
$$= -\text{"dim"}\,\Omega^0(\mathrm{ad}\,P) + \text{"dim"}\,\Omega^1(\mathrm{ad}\,P) - \text{"dim"}\,\Omega^+(\mathrm{ad}\,P)$$
$= -$index of the AHS complex
$$\Omega^0(\mathrm{ad}\,P) \xrightarrow{d_A} \Omega^1(\mathrm{ad}\,P) \xrightarrow{d_A^+} \Omega^+(\mathrm{ad}\,P)$$

This is nothing but the "virtual" dimension of the moduli space of ASD connections.

$\langle$ some cohomology class, [moduli] $\rangle$: Donaldson invariant
(after lots of technicalities)

We can formally express the Euler class by a path integral as the Chern–Weil theory for ∞ rank vector bundles.

Donaldson inv. = a correlation function of a "twisted" version of $N=2$ SUSY Yang–Mills theory.

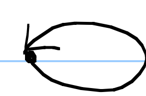
Suppose $X = Y^3 \times \mathbb{R}$ with $g = g_Y \otimes dt^2$.
$(y\ ,\ t)$

In temporary gauge (i.e. $A_t = 0$),
$A = A_\alpha(y,t)\,dy^\alpha$ is a 1-parameter family of connections on Y.

Prop. A gradient flow of the Chern-Simons functional on $\mathcal{A}_Y$
$=$ an anti-self-dual connection on $Y \times \mathbb{R}$.

This is a starting point of the definition of the (Instanton) Floer homology.

Since CS is not well-defined on $\mathcal{B}_Y = \mathcal{A}_Y / \mathcal{G}_Y$, it is possible that a gradient flow connects a critical point to itself.



We take $Y = S^3$ hereafter
$$X = S^3 \times \mathbb{R} \underset{\text{conformal}}{\sim} \mathbb{R}^4 \setminus 0$$

We consider finite action ASD connections on $\mathbb{R}^4 \setminus 0$.
$\|F_A\|_{L^2} < \infty$

$\mathbb{R}^4 \setminus 0 \overset{\text{conf.}}{\sim} S^4 \setminus \{n.p, s.p.\}$

Th (Uhlenbeck)
A finite action ASD conn. on $\mathbb{R}^4 \setminus 0$ extends to S^4.

$\mathcal{M}(k, G)$ = framed moduli space of ASD connections on S^4
= gauge equiv. classes of ASD connections together with a triivalition $\varphi : P|_\infty \cong G$

Here $k = \frac{1}{8\pi^2} \int_{S^4} p_1(\mathrm{ad}\, P)$: instanton number.

Th. $\mathcal{M}(k.G)$ is a hyperKähler manifold of $\dim_{\mathbb{R}} = 4k h^\vee$ ($h^\vee$ = dual Coxeter #)

$\mathcal{M}(k.G)$ is not compact, due to a) bubbling.
b) translation symmetry

Let us kill a).
Uhlenbeck (partial) compactification :

$$\overline{\mathcal{M}}(k.G) = \coprod_{n \leq k} \mathcal{M}(n, G) \times S^{k-n} \mathbb{R}^4$$

topology : $A_i \to (A_\infty, x_1 + \cdots + x_{k-n})$

$\Leftrightarrow$ $|F_{A_i}|^2 dvol \to |F_{A_\infty}|^2 dvol + 8\pi^2 \sum_i \delta_{x_i}$

as measure

Rem. If we define the same space on a compact manifold X, then we get an actual compactification.

Th. $\overline{M}(k, G)$, together with any of cpx str. on $M(k, G)$, has a structure of an affine scheme.

(Braverman - Finkelberg - Gaitsgory Quasimaps into affine Grassman
Biswas more general construction

- ADHM description

$G = SU(r)$

$M(k, G) = (\mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(0))^{reg} / U(k) = \mu_{\mathbb{C}}^{-1}(0)^{st. \& cost.} / GL(k)$

$\overline{M}(k, G) = \mu_{\mathbb{C}}^{-1}(0) /\!/ GL(k)$

$$\mathbb{M} = \{ (B_1, B_2, i, j) \mid \begin{array}{l} B_\alpha \in End(\mathbb{C}^k) \\ i \in Hom(\mathbb{C}^r, \mathbb{C}^k) \\ j \in Hom(\mathbb{C}^k, \mathbb{C}^r) \end{array} \}$$

$$\begin{tikzcd} \mathbb{C}^k \arrow[loop left, "B_1"] \arrow[loop right, "B_2"] \arrow[d, shift right, "j"'] \\ \mathbb{C}^r \arrow[u, shift right, "i"'] \end{tikzcd}$$

$\mu_{\mathbb{C}}(B_1, B_2, i, j) = [B_1, B_2] + ij$

$$\mu_{\mathbb{R}}(B_1,B_2,i,j)=\frac{\sqrt{-1}}{2}\left([B_1,B_1^*]+[B_2,B_2^*]+ii^*-j^*j\right)$$

stable $\Leftrightarrow$ $\not\exists\, S\subsetneq\mathbb{C}^k$ $\quad B_\alpha(S)\subset S$, $\mathrm{Im}\, i\subset S$

costable $\Leftrightarrow$ $({}^tB_\alpha, {}^tj, {}^ti)$ is stable

In particular, $\mathbb{C}[\overline{M}(k,G)]=\mathbb{C}[\mu_{\mathbb{C}}^{-1}(0)]^{GL(k)}$.

<u>Rem.</u> $\exists$ $SO(n), Sp(n)$-versions

○ partition function

$T\subset G$: maximal torus

$\tilde{T}=S^1\times S^1\times T$

$\tilde{T}\curvearrowright \overline{M}(k,G)$ $\qquad T$: change of the framing

$S^1\times S^1\curvearrowright \mathbb{R}^4\cong\mathbb{C}^2$ $\quad (t_1x, t_2y)$

<u>Lemma</u> $\overline{M}(k,G)^{\tilde{T}}=\{(\text{triv. ASD connection}, k\cdot 0)\}$

$\in M(0,G)\times S^k\mathbb{R}^4\subset\overline{M}(k,G)$

(proof) ① $\mathbb{R}^4\curvearrowleft S^1\times S^1$ $\rightsquigarrow$ the fixed point $=\{0\}$

② A : ASD conn. is fixed by T

$\Leftrightarrow$ the structure group can be reduced from G to T.

But there is no nontrivial $U(1)$-instanton,

as $p_1=0$ automatically. //

Recall localization thm for equiv. homology :

$$H_*^{\widetilde{T}}(\overline{M}(k,G)) \otimes_{S(\widetilde{t}^*)} \mathcal{S}(\widetilde{t}^*) \underset{i_*}{\overset{\cong}{\longleftarrow}} H_*^{\widetilde{T}}(\overline{M}(k,G)^{\widetilde{T}}) \otimes_{S(\widetilde{t}^*)} \mathcal{S}(\widetilde{t}^*)$$

where $S(\widetilde{t}^*)$ = symmetric algebra of $\widetilde{t}^* = (\mathrm{Lie}\,\widetilde{T})^*$

$= \mathbb{C}[\varepsilon_1, \varepsilon_2, a_1, \cdots, a_r]$ $\quad r = \mathrm{rank}\, G$

$\mathcal{S}(\widetilde{t}^*)$ = its quotient field

(For $G = SU(r)$, we usually take $a_1, \cdots, a_r$ with the constraint $a_1 + \cdots + a_r = 0$

By the above lemma, $\mathrm{RHS} = \mathcal{S}(\widetilde{t}^*)$

Rem. $(i_*)^{-1}$ is, in general, hard to compute. But for a smooth mfd,

$$(i_*)^{-1} = \sum_p \frac{i_p^*}{e(T_pM)}$$

(Toy example) $[\mathbb{C}^2] = \dfrac{[0]}{e(T_0\mathbb{C}^2)} = \dfrac{1}{\varepsilon_1\varepsilon_2}$

Def. (Nekrasov)

The instanton part of the deformed partition function :

$$Z_G^{\mathrm{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) = \sum_{k=0}^{\infty} \Lambda^{2h^\vee k} [\overline{M}(k,G)] \in \mathcal{S}(\widetilde{t}^*)[[\Lambda]]$$

○ Variants :

- 5D version

replace $H_*^{\widetilde{T}}$ by $K^{\widetilde{T}}$.

$\mathcal{R}(\widetilde{T})$ = quot. field of $R(\widetilde{T})_{\otimes \mathbb{C}} = \mathbb{C}(e^{\beta\varepsilon_1}, e^{\beta\varepsilon_2}, e^{\beta a_\alpha})$

β : formal param. $\beta \to 0$: 4D

Then $[\mathcal{O}_{\overline{M}(k,G)}] \in K^{\widetilde{T}}(\overline{M}(k,G)) \otimes_{R(\widetilde{T})} \mathcal{R}(\widetilde{T}) \cong \mathcal{R}(\widetilde{T})$

is, in fact, the character of the coordinate ring :

$ch_{\widetilde{T}}\, \mathbb{C}[\overline{M}(k,G)] \in \mathbb{Z}_{\geq 0}[[e^{\beta\varepsilon_1}, e^{\beta\varepsilon_2}, e^{\beta a_1}, \cdots, e^{\beta a_r}]]$

(wt spaces are finite dimensional)

<u>Rem</u>. We need to correct the above by $K_{\overline{M}(k,G)}^{1/2}$ (Dirac op. v.s. $\bar\partial$-op.)

($K_{\overline{M}}$ is trivial, but is nontrivial as char.)

- with matters

$\rho : G \to U(M)$ a representation

$G(M)$ = centralizer of ρ : matter group $\supset T(M)$ torus

$P \times_G M$: associated vector bundle

For an ASD connection A, we consider

$$D\!\!\!/_A^M : \Gamma(S^- \otimes P \times_G M) \to \Gamma(S^+ \otimes P \times_G M)$$

$\operatorname{Ker} D\!\!\!/_A^M$: vector bundle over $M(k,G)$

naturally $\widetilde{T} \times T(M)$ equivariant

$$\Rightarrow \quad e(\mathrm{Ker}\, \cancel{D}_A^M) \cap [M(k,G)] \in H_*^{\tilde{T}\times T(M)}(M(k,G))$$

This extends to $H_*^{\tilde{T}\times T(M)}(\overline{M}(k,G))$

at least for G: classical group thanks to ADHM.

But it is not clear for me,

whether we have a canonical extension.

For $G = SU(r)$, and

- fundamental matter : $\rho =$ ~~copies of~~ direct sum of vector repr.

- adjoint matter : $\rho =$ adjoint repr.

we have "canonical" extensions.

(see below)

<u>Rem.</u> degree of $e(\mathrm{Ker}\, \cancel{D}_A^M) \cap [M(k,G)]$

$$= 4k(h^\vee - C_2(\rho)) \qquad h^\vee = C_2(ad)$$

The "feature" of the theory is different according to

$h^\vee - C_2(\rho) > 0$ ---- asymptotically free

$h^\vee - C_2(\rho) = 0$ ----- critical similar to $N=4$ SYM

$<$? people don't consider usually.

○ $G = SU(r)$ case (Gieseker - Maruyama cptification)

$\overline{M}(k, SU(r))$ has a nice resolution of singularities :

$M(k,r) =$ framed torsion-free sheaves E on $\mathbb{P}^2$

& $\varphi : E|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}^{\oplus r}$

$$M(k,r) \longrightarrow \overline{M}(k, SU(r)) = \coprod_{\ell} M(\ell, SU(r)) \times S^{k-\ell}\mathbb{C}^2$$

$$E \longmapsto (E^{\vee\vee}, \operatorname{Supp} E^{\vee\vee}/E)$$

locally-free $\quad \exists 1$ ASD conn. by Hitchin-Kobayashi

or $M(k,r) = \mu_{\mathbb{C}}^{-1}(0)^{stable}/GL(k) \longrightarrow \mu_{\mathbb{C}}^{-1}(0)/\!\!/GL(k) = \overline{M}(k, SU(r))$

- $\operatorname{Ker} \mathcal{D}_A^{\text{fund.}} \cong H^1(\mathbb{P}^2, E(-\ell_\infty)) \quad (H^0, H^2 = 0)$

extends to $M(k,r)$.

- $\operatorname{Ker} \mathcal{D}_A^{\text{adj.}} \cong \operatorname{Ext}^1(E, E(-\ell_\infty)) \qquad \operatorname{Ext}^{0,2} = 0$

tangent space

$$\pi : M(k,r) \longrightarrow \overline{M}(k, SU(r))$$

$$\pi_*[M(k,r)] = [\overline{M}(k, SU(r))]$$

Moreover,

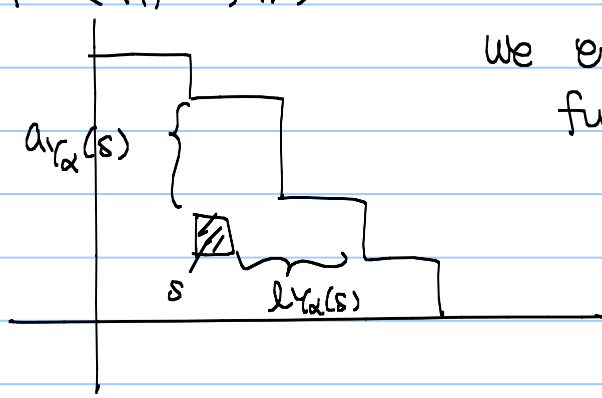
$$M(k,r)^{\tilde{T}} = \{ E = I_1 \oplus \cdots \oplus I_r \mid I_\alpha : \text{monomial ideal} \}$$

$I_\alpha \leftrightarrow Y_\alpha$: Young diagram

$$\pi_*[M(k,r)] = \sum_{p \in M(k,r)^{\tilde{T}}} \frac{1}{e(T_p M(k,r))}$$

This gives us a combinatorial expression of the partition function.

$\vec{Y} = (Y_1, \cdots, Y_r)$



We extend Macdonald leg/arm functions to a box outside Y

$e(T_p M(k,r))$

$$= \prod_{\alpha,\beta} \prod_{s \in Y_\alpha} \left(a_\beta - a_\alpha - \varepsilon_1 l_{Y_\beta}(s) + \varepsilon_2 (a_{Y_\alpha}(s)+1) \right)$$

$$\times \prod_{t \in Y_\beta} \left(a_\beta - a_\alpha + \varepsilon_1 (l_{Y_\alpha}(t)+1) - \varepsilon_2 a_{Y_\beta}(t) \right)$$

$$* \quad \mathrm{Ext}^1(E, E(-l_\infty)) = \bigoplus_{\alpha,\beta} \mathrm{Ext}^1(I_\alpha, I_\beta(-l_\infty))$$

We can further consider the correlation functions:

$\mathcal{E} \longrightarrow \mathbb{P}^2 \times M(k,r)$ universal sheaf
(canonically exists
thanks to the framing)

$\mathrm{ch}_{p+1}(\mathcal{E})/[\mathbb{C}^2]$: the formal slant product

$$[\mathbb{C}^2] = \frac{1}{\underbrace{\varepsilon_1 \varepsilon_2}_{= e(T_0 \mathbb{C}^2)}} [0]$$

$Z^{inst}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\tau}, \Lambda)$

$$= \sum_{k=0}^{\infty} \Lambda^{4kr} \pi_* \left[\exp\left(\sum_{p=1}^{\infty} \tau_p \, \mathrm{ch}_{p+1}(\mathcal{E})/[\mathbb{C}^2] \right) \cap [M(k,r)] \right]$$

- 5D Chern-Simons term

$$\mathcal{L} = \det H^1(\mathbb{P}^2, \mathcal{E}(-\ell_\infty))$$

We replace $[\mathcal{O}_{\overline{M}(k, SU(r))}]$ by

$$\pi_*(\mathcal{L}^{\otimes m}) \qquad m \in \mathbb{Z}$$

i.e. $\mathrm{ch}_{\mathbb{T}} \sum_i (-1)^i H^i(M(k,r), \mathcal{L}^{\otimes m})$

Physicists say only $|m| \leq r$.

This is consistent with

- SW curve
- geometric engineering

But I do not understand the gauge theoretic explanation.

Rem. This is explained as a positivity of the "β-function".

In 4D theory, the β-function is related to the virtual dimension of the homology cylce, but in 5D theory, it is more subtle......

○ geometric engineering

G : simply-laced $\leftrightsquigarrow$ $\Gamma \subset SU_2$ finite subgroup

McKay : $H_2(\widetilde{\mathbb{C}^2/\Gamma}) \cong \mathfrak{h}^*$: Cartan subalg

$\cup$

$[C_i] \longleftrightarrow \alpha_i$ simple root

$X_\Gamma = \widetilde{\mathbb{C}^2/\Gamma}$: ALE space in the literature.

<u>Rem</u>. The connection between $G \leftrightarrow \Gamma$ can be deepened various ways :

H_{mid}(moduli of ASD conn's on X_Γ with some bdry cond.)

$=$ integrable highest weight rep. of $\hat{\mathfrak{g}}$

$\mathfrak{X}_\Gamma$: X_Γ family over $\mathbb{P}^1$

$\downarrow$ resolution of $\mathcal{O}(-1) \otimes \mathbb{C}^2/\Gamma$

$\mathbb{P}^1$ $\downarrow$

$\mathbb{P}^1$

$H_2(\mathfrak{X}_\Gamma) \ni [\text{base } \mathbb{P}^1]$, $[C_i]$

$i = 1, \cdots, r$

$d = d_b \cdot [\mathbb{P}^1] + d_i [C_i]$

$$F^{d_b>0}_{GW}(g_s, \vec{a}, \Lambda)$$

$$= \sum_{\substack{g\geq 0, d \\ d_b>0}} [M_{g,d}(\underset{\underset{Q}{\circlearrowleft}}{\mathcal{X}_\Gamma})]^{vir} \cdot g_s^{2g-2} \left(\frac{\beta\Lambda}{2}\right)^{d_b} e^{\beta a_i d_i}$$

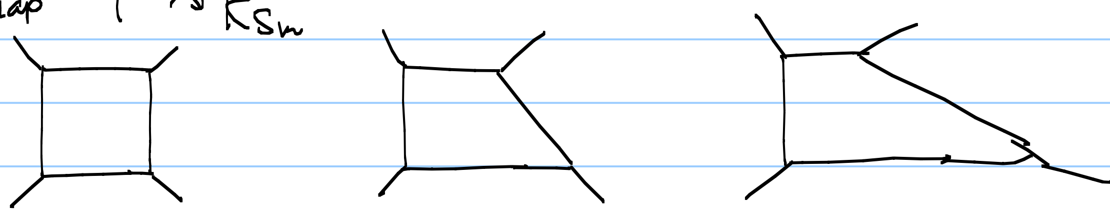
$$F^{d_b>0}_{GW}(g_s, \vec{a}, \Lambda) = \log Z^{inst}_{5D}(\underset{\varepsilon_1}{\underset{\shortparallel}{g_s}}, \underset{\varepsilon_2}{\underset{\shortparallel}{-g_s}}, \vec{a}, \Lambda)$$

We still need to specify $\mathcal{X}_\Gamma$.

$r=2$ case $\mathbb{P}^1$-family over $\mathbb{P}^1$ ... Hirzebruch surface S_m

$\therefore\ \mathcal{X}_{A_1} = K_{S_m}$ nef : $m = 0, 1, 2$

moment map image $T^2 \curvearrowright K_{S_m}$



5D instanton partition function with $CS = m$

$$K_{S_0} \longrightarrow \mathcal{O}(-1)\oplus\mathcal{O}(-1) \Big/ \mathbb{Z}_2$$
$$\downarrow$$
$$\mathbb{P}^1$$

crepant resolutions

$$K_{S_2} \longrightarrow \mathcal{O}(-2)\oplus\mathcal{O} \Big/ \mathbb{Z}_2$$
$$\downarrow$$
$$\mathbb{P}^1$$

$$K_{S'} \longrightarrow \Big\{ \underset{X}{\mathcal{O}(-1)} \oplus \underset{Y}{\mathcal{O}(-2)} \oplus \underset{Z}{\mathcal{O}(-3)} \;\Big|\; Y^2 = XZ \Big\}$$
$$\downarrow$$
$$\mathbb{P}^1$$